

## Note

## Computational complexity of some restricted instances of 3-SAT

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Abstract

Tovey [A simplified satisfiability problem, *Discrete Appl. Math.* 8 (1984) 85–89] showed that it is NP-hard to decide the satisfiability of 3-SAT instances in which every variable occurs four times, while every instance of 3-SAT in which each variable occurs three times is satisfiable. We explore the border between these two problems. Answering a question of Iwama and Takaki, we show that, for every fixed  $k \geq 0$ , there is a polynomial-time algorithm to determine the satisfiability of 3-SAT instances in which  $k$  variables occur four times and the remaining variables occur three times. On the other hand, it is NP-hard to decide the satisfiability of 3-SAT instances in which all but one variable occurs three times, and the remaining variable is allowed to occur an arbitrary number of times.

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1. Introduction

An instance of  $k$ -SAT is a set of clauses that are disjunctions of exactly  $k$  literals. The problem is to determine whether there is an assignment of truth values to the variables such that all the clauses are satisfied. It is well known that 2-SAT can be solved in polynomial time, while Cook [4] showed that  $k$ -SAT is NP-hard for  $k \geq 3$ . This leads to the general question of exploring the boundary region between polynomial-time and NP-hard satisfiability problems, by considering more or less restricted problem instances (of course, this is most interesting if  $P \neq NP$ ).

One way to restrict instances of  $k$ -SAT is to limit the number of times a variable can occur. Tovey [10] showed that instances of 3-SAT in which every variable occurs three times are always satisfiable (this is an immediate corollary of Hall's Theorem), while it is NP-hard to decide the satisfiability of 3-SAT instances in which every variable occurs four times. Results of this type for  $k > 3$  were obtained by Dubois [5] and Kratochvíl et al. [7], and also in [2,3]. The approximation hardness of the corresponding maximization problem was studied in [2,3].

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The boundary between three occurrences and four occurrences of variables in 3-SAT was further examined by Iwama and Takaki [6]. Let us write  $(3, 4^{(k)})$ -SAT for the set of 3-SAT instances in which  $k$  variables occur four times and the remaining variables occur three times, and  $(3, 4^{(k)}, n)$ -SAT for the set of instances of  $(3, 4^{(k)})$ -SAT with  $n$  variables. Thus  $(3, 4^{(0)})$ -SAT is the collection of 3-SAT instances in which every variable occurs exactly three times. Note that  $k$  must be divisible by three, as the total number of literals is three times the number of clauses. Iwama and Takaki showed that every instance of  $(3, 4^{(3)})$ -SAT is satisfiable, while there are unsatisfiable instances of  $(3, 4^{(9)})$ -SAT. They further asked whether there is a constant  $k$  such that  $(3, 4^{(k)})$ -SAT is NP-hard. In this paper, we give a negative answer to this question (under the  $P \neq NP$  assumption). We also show that it is NP-hard to decide the satisfiability of 3-SAT instances in which all but one variable occurs three times, and the remaining variable is allowed to occur an arbitrary number of times.

We remark that there are other interesting ways to explore the border between polynomial-time and NP-hard satisfiability problems. For instance, instead of looking at restrictions of 3-SAT, one can consider extensions of 2-SAT. This line of investigation has been pursued by several authors, including Monasson and Zecchina [8], Monasson et al. [9], Anderson [1], and Zhao et al. [11].

## 2. Results

We begin by answering the question raised by Iwama and Takaki.

**Theorem 1.** *Satisfiability of instances of  $(3, 4^{(k)}, n)$ -SAT can be determined in time  $2^{k/3} n^{k/3} \text{poly}(n)$ .*

Thus for any fixed  $k$ ,  $(3, 4^{(k)}, n)$ -SAT instances can be solved in polynomial time.

In order to prove Theorem 1, we shall rely on the fact that satisfiable instances of  $(3, 4^{(k)})$ -SAT have satisfying assignments with a particular structure. Let  $I$  be a satisfiable instance of  $(3, 4^{(k)})$ -SAT with clauses  $\mathcal{C}$  and variables  $V$ , and let  $\phi$  be a satisfying assignment. A *witness function*  $w : \mathcal{C} \rightarrow V$  for  $\phi$  is a function that, for each  $C \in \mathcal{C}$ , chooses a variable  $w(C)$  which occurs (possibly negated) as a literal in  $C$  evaluating to true under  $\phi$ . Thus  $w(C)$  is a variable that “witnesses” the satisfaction of  $C$  in  $\phi$ .

Note that if  $w$  is a witness function for some satisfying assignment and  $w(C_1) = w(C_2) = x$  then  $x$  must occur as a literal with the same sign in  $C_1$  and  $C_2$ . On the other hand, any function satisfying this condition can be used to find a satisfying assignment of  $I$ . We shall call such a function *consistent*.

**Lemma 1.** *If  $I = (\mathcal{C}, V)$  is a satisfiable instance of  $(3, 4^{(k)})$ -SAT then there is a satisfying assignment  $\phi$  with a surjective witness function  $w : \mathcal{C} \rightarrow V$ .*

**Proof.** Let  $(\phi, w)$  be a satisfying assignment and a witness function chosen such that the size of the image of  $w$  is maximal (over all such pairs). If  $w$  is surjective then we are done. Otherwise, let us consider the bipartite graph  $G$  with vertex classes  $\mathcal{C}$  and  $V$  and an edge from  $C \in \mathcal{C}$  to  $v \in V$  if and only if  $v$  occurs (with either sign) in  $C$ . We shall say that an edge  $vC$  is *used* if  $w(C) = v$ ; otherwise  $vC$  is *unused*.

Let  $U = \{v \in V : |w^{-1}(v)| \geq 2\}$  be the set of variables that are used as witnesses by more than one clause (note that, since  $w$  is not surjective,  $|U| > 0$ ). Let  $s := \sum_{v \in U} |w^{-1}(v)| \geq 2|U|$  be the number of used edges incident with  $U$ . An *alternating path* in  $V$  is a path  $v_1 C_1 v_2 C_2 \dots v_k$  or  $v_1 C_1 v_2 \dots C_k$  for some  $k \geq 1$  such that  $v_1 \in U$ , all edges  $v_i C_i$  are used, and all edges  $C_i v_{i+1}$  are unused. We shall show that if  $w$  is not surjective then we can use a suitable alternating path to construct a new witness function  $w'$  with a larger image.

Let us say that a vertex of  $G$  is *reachable* if it belongs to any alternating path. Let  $U'$  be the set of reachable vertices in  $V \setminus U$ , and let  $\mathcal{B}$  be the set of reachable vertices in  $\mathcal{C}$ . Note that if  $u \in U \cup U'$  and the edge  $uC$  is used then it is easy to find an alternating path that contains  $C$  (take an alternating path as far as  $u$ , and if  $C$  has not already been visited then extend the path with the edge  $uC$ ); it follows that  $C \in \mathcal{B}$ .

There are  $s$  used edges incident with  $U$  and  $|U'|$  used edges incident with  $U'$ . Since every clause is incident with exactly one used edge, we have

$$|\mathcal{B}| \geq s + |U'|. \quad (1)$$

On the other hand, each clause is incident with at least two unused edges, and so there are at least  $2|\mathcal{B}|$  unused edges incident with  $\mathcal{B}$ . It is easy to check that if  $C \in \mathcal{B}$  and  $vC$  is unused then  $v \in U \cup U'$  (we argue as before: take an alternating path as far as  $C$  and add the edge  $Cv$  if necessary). Thus there are at least  $2|\mathcal{B}|$  unused edges incident with  $U \cup U'$ .

Now suppose that every vertex of  $U \cup U'$  is in the image of  $w$ . Since the vertices of  $U \cup U'$  are incident with  $s + |U'|$  used edges, and at most  $k$  vertices have degree 4, it follows that the number of unused edges incident with  $U \cup U'$  is at most

$$3(|U| + |U'|) + k - (s + |U'|) = 3|U| + 2|U'| + k - s.$$

Since this is at least  $2|\mathcal{B}|$ , it follows from (1) that

$$2(s + |U'|) \leq 3|U| + 2|U'| + k - s,$$

and so

$$s \leq |U| + k/3. \quad (2)$$

Now let  $U'' = \text{Im}(w) \setminus U$ . Since  $U \subset \text{Im}(w)$  and  $\text{Im}(w) \neq V$ , we have

$$|U| + |U''| \leq |V| - 1. \quad (3)$$

Since every vertex of  $U''$  is incident with exactly one used edge, and there are  $|\mathcal{C}| = |V| + k/3$  used edges in total, we have

$$s + |U''| = |V| + k/3. \quad (4)$$

Therefore, by (3) and (4),

$$s = |V| + k/3 - |U''| \geq |U| + k/3 + 1$$

which contradicts (2).

It follows that there is some vertex  $u' \in U'$  that is not contained in the image of  $w$ . Let  $P = v_1 C_1 v_2 \dots v_k C_k u'$  be a shortest alternating path from  $U$  to  $u'$ : note that  $v_1$  is in  $U$ , but  $v_i \notin U$  for  $i > 1$ . Exchanging used and unused edges in  $P$ , we obtain a consistent witness function  $w'$  with a larger image than  $w$ . This contradicts the maximality of the image of  $w$ , and we therefore deduce that  $w$  is surjective.  $\square$

We can now prove Theorem 1.

**Proof of Theorem 1.** If  $I$  is satisfiable then there are a satisfying assignment  $\phi$  and a surjective witness function  $w$  for  $\phi$ . Since there are  $n + k/3$  clauses, and  $n$  variables, it follows that at most  $k/3$  variables are covered more than once by  $w$ . We can therefore search for such a  $w$  by explicitly examining every set of  $\lfloor k/3 \rfloor$  variables and every assignment of those variables, and then checking for a matching from the remaining unsatisfied clauses to the remaining unassigned variables.  $\square$

We have shown that for any constant,  $(3, 4^{(k)})$ -SAT instances can be solved in polynomial time. What if we allow more than four occurrences of some variables? The following theorem shows that if we allow an unbounded number of occurrences of even one variable, then the problem becomes NP-hard.

**Theorem 2.** *The restriction of 3-SAT to the set of instances in which all but one variable occur exactly three times is NP-hard.*

We shall prove Theorem 2 by reduction from another problem. We define  $(O3, L \leq 3)$ -SAT to be the set of instances of satisfiability in which every variable occurs three times and every clause has length at most 3. This problem is NP-hard, as was shown by Tovey [10]. We give a proof for completeness.

**Theorem 3.** *Determining the satisfiability of instances of  $(O3, L \leq 3)$ -SAT is NP-hard.*

**Proof.** We give a reduction from 3-SAT due to Tovey [10]. Given an instance of 3-SAT, we run through its variables in turn, modifying the instance as follows. If a variable  $x$  occurs at most three times we do nothing. If  $x$  occurs  $d > 3$  times, we introduce new variables  $x_1, \dots, x_d$  and 2-clauses  $x_1 \vee \neg x_2, \dots, x_{d-1} \vee \neg x_d, x_d \vee \neg x_1$ . We then replace the  $d$  occurrences of  $x$  by  $x_1, \dots, x_d$  in turn and remove  $x$ . It is easily checked that this preserves satisfiability/unsatisfiability, and when we have dealt with all the variables we have an equivalent instance of  $(O3, L \leq 3)$ -SAT.  $\square$

Note that a typical variable in the instance constructed above will belong to two clauses of length 2. We sketch a slightly more complicated construction that allows us to insist that every variable belongs to at most *one* clause of length 2. Given an instance of 3-SAT, we first perform the construction in the proof above to obtain an instance  $I$  of  $(O3, L \leq 3)$ -SAT. We introduce new variables  $a_i, b_i, \dots, i = 1, 2, 3$ . We build a new instance as follows: take the clause  $\neg a_1 \vee \neg a_2 \vee \neg a_3$ , and for each  $i$  add the following chain of clauses:

$$\begin{array}{llll} a_i & b_i & c_i & \\ a_i & \neg b_i & & \\ b_i & \neg c_i & d_i & \\ & \neg d_i & e_i & \\ & \neg d_i & \neg e_i & f_i \\ & & \vdots & \end{array}$$

For each  $i$ , take a copy  $I_i$  of  $I$  (on a new set of variables), and extend all of its 2-clauses to 3-clauses by adding the negation of some variable that occurs twice in the corresponding chain (i.e. one of  $\{\neg c_i, \neg e_i, \dots\}$ ). This gives the required instance. Note that in any satisfying assignment of the resulting instance, one of the variables  $a_i$  must be false, and so the variables  $c_i, e_i, \dots$  must be true, which means that  $I_i$  (and hence  $I$ ) is satisfiable. On the other hand, if  $I$  is satisfiable, then the new instance is easily seen to be satisfiable.

We now return to the proof of Theorem 2.

**Proof.** We give a reduction from  $(O3, L \leq 3)$ -SAT. Let  $I$  be an instance of  $(O3, L \leq 3)$ -SAT. We take two copies  $I_1, I_2$  of  $I$  (on disjoint sets of variables) and a new variable  $x$ . We construct an instance of 3-SAT by adding  $x$  to every 2-clause in  $I_1$  and  $\neg x$  to every 2-clause in  $I_2$ . The resulting instance is clearly equivalent to  $I$ .  $\square$

### 3. Conclusion

We have given a polynomial-time algorithm for instances of 3-SAT in which a constant number of variables occur four times and the remainder occur three times. What happens if we allow the number of variables occurring four times to grow slowly?

**Problem.** *Is there a function  $f(n)$  that tends to  $\infty$  as  $n \rightarrow \infty$  such that the satisfiability of instances of  $(3, 4^{(f(n))}, n)$ -SAT is solvable in polynomial time? (We assume here that  $f(n)$  is a multiple of three, to exclude trivial cases.)*

We conjecture that for sufficiently slow-growing  $f$  the problem can be solved in polynomial time.

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